Auxiliary Differential Equation Formulation: An Efficient Implementation of the Perfectly Matched Layer

Omar Ramadan, *Member, IEEE*

***Abstract*—An efficient algorithm for implementing the perfectly matched layer (PML) is presented for truncating finite-difference time-domain domains. The algorithm is based on incorporating the auxiliary differential equation method into the PML formulations. Simple, unsplit-field and material independent PML formulations are obtained. Two dimensional numerical examples are included to validate the proposed formulations.**

***Index Terms*—Auxiliary differential equation (ADE), finite-difference time-domain (FDTD), perfectly matched layer (PML).**

I. Introduction

The perfectly matched layer (PML), introduced by Berenger [1], has been widely used for truncating finite-difference time-domain (FDTD) domains. However, Berenger's PML is based on splitting each of the field components of Maxwell's equations into two subcomponents and it is only applies, as originally proposed, for truncating nonconductive media. For conductive media, alternative PML formulations based on the stretched coordinate approach have been introduced [2]. But, the FDTD implementation of these formulations necessitate similar splitting of the field components. Recently, unsplit field implementation of the stretched coordinate PML formulations has been introduced [3]. This method, which is named as convolutional PML (CPML), is based on applying the convolution theorem into the stretched coordinate PML formulations [3].

In this letter, alternative and simple unsplit-field implementations of the stretched coordinates PML formulations are introduced without the need of computing the convolution terms introduced in [3]. These formulations are based on incorporating the auxiliary differential equation (ADE) method into the PML implementations. For truncating general media, it is shown that this method, as in the case of the CPML, requires two additional auxiliary variables per field component per cell in the PML region. In addition, the presented formulations are implemented using the magnetic field $\mathbf{H}$ and the electric displacement field $\mathbf{D}$ instead of the electric field $\mathbf{E}$ [4]. This allows the PML to be independent of the material of the FDTD computational domain [4]. Two dimensional numerical tests have been carried out to validate the proposed method.

Manuscript received April 26, 2002; revised August 2, 2002. The review of this letter was arranged by Associate Editor Dr. Shigeo Kawasaki.

The author is with the Computer Engineering Department, Eastern Mediterranean University, Gazi Magusa, Mersin, Turkey.

Digital Object Identifier 10.1109/LMWC.2003.808706

II. Formulation

Using the stretched coordinate formulations [2], the frequency domain modified Maxwell's curl equations in the PML region can be written as

$$\nabla_s \times \mathbf{H} = j\omega\,\mathbf{D} \tag{1}$$

$$\nabla_s \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{2}$$

where the operator ∇_s is expressed as

$$\nabla_s = \hat{a}_x S_x^{-1}\partial x + \hat{a}_y S_y^{-1}\partial y + \hat{a}_z S_z^{-1}\partial z \tag{3}$$

where ∂x, ∂y, and ∂z are the space derivatives with respect to x, y and z and S_η, $(\eta = x, y, z)$, are called the stretched coordinate variables. To achieve good electromagnetic wave absorption, S_η, $(\eta = x, y, z)$, are chosen within the PML region [2] as

$$S_\eta = 1 + \sigma_\eta / j\omega\varepsilon_o, \qquad (\eta = x, y, z) \tag{4}$$

where σ_η is the conductivity profile along the η-direction in the PML region.

In the above formulations, the electric field components are computed from the relation $\mathbf{D} = \varepsilon_o \hat{\varepsilon}_r(\omega)\mathbf{E}$ where $\hat{\varepsilon}_r(\omega)$ is the complex relative permittivity of the FDTD computational domain. This approach has the advantage of making the PML to be independent of the material of the FDTD computational domain [4].

To descretize (1) and (2), consider, as an example, the D_z-field component of (1)

$$j\omega D_z = S_x^{-1}\partial x H_y - S_y^{-1}\partial y H_x. \tag{5}$$

Using the partial fraction expansion, $S_\eta^{-1}(\eta = x, y)$ can be written as

$$S_\eta^{-1} = j\omega/(j\omega + \sigma_\eta/\varepsilon_o) = 1 - (\sigma_\eta/\varepsilon_o)/(j\omega + \sigma_\eta/\varepsilon_o). \tag{6}$$

Substituting (6) into (5), we obtain

$$j\omega D_z = \partial x H_y - f_{zx} - \partial y H_x + f_{zy} \tag{7}$$

where the auxiliary variables f_{zx} and f_{zy} are given by

$$f_{zx} = (\sigma_x/\varepsilon_o)\partial x H_y/(j\omega + \sigma_x/\varepsilon_o) \tag{8}$$

$$f_{zy} = (\sigma_y/\varepsilon_o)\partial y H_x/(j\omega + \sigma_y/\varepsilon_o). \tag{9}$$

Equations (8) and (9) can be written as

$$j\omega f_{zx} + (\sigma_x/\varepsilon_o) f_{zx} = (\sigma_x/\varepsilon_o)\partial x H_y \tag{10}$$

$$j\omega f_{zy} + (\sigma_y/\varepsilon_o) f_{zy} = (\sigma_y/\varepsilon_o)\partial y H_x. \tag{11}$$

Transforming (7) into the time domain using the Fourier transform relation $j\omega \longrightarrow \partial t$, where ∂t is the derivative with respect to time, we obtain

$$\partial t D_z = \partial x H_y - f_{zx} - \partial y H_x + f_{zy} \tag{12}$$

where the auxiliary variables f_{zx} and f_{zy} are given by the following first order differential equations obtained by transforming (10) and (11) into the time domain

$$\partial t f_{zx} + (\sigma_x/\varepsilon_o) f_{zx} = (\sigma_x/\varepsilon_o)\partial x H_y \tag{13}$$

$$\partial t f_{zy} + (\sigma_y/\varepsilon_o) f_{zy} = (\sigma_y/\varepsilon_o)\partial y H_x. \tag{14}$$

It is interesting to note that by equating these auxiliary variables to zero, (12) will be reduced to the standard Yee's FDTD formulations. Following Yee's FDTD algorithm for discretizing the space and the time derivatives in (12) [2], the following FDTD expression to calculate the D_z field can be obtained

$$\begin{aligned} D^{n+1}_{z_{i,j,k+1/2}} &= D^{n}_{z_{i,j,k+1/2}} + \Delta t\left(H^{n+1/2}_{y_{i+1/2,j,k+1/2}} - H^{n+1/2}_{y_{i-1/2,j,k+1/2}}\right)\Big/\Delta x \\ &\quad - \Delta t\left(f^{n+1}_{zx_{i,j,k+1/2}} + f^{n}_{zx_{i,j,k+1/2}}\right)\Big/ 2 \\ &\quad - \Delta t\left(H^{n+1/2}_{x_{i,j+1/2,k+1/2}} - H^{n+1/2}_{x_{i,j-1/2,k+1/2}}\right)\Big/\Delta y \\ &\quad + \Delta t\left(f^{n+1}_{zy_{i,j,k+1/2}} + f^{n}_{zy_{i,j,k+1/2}}\right)\Big/ 2 \end{aligned} \tag{15}$$

where Δt is the time step, Δx and Δy are the space cell size in the x and y directions, respectively, and $f^{n+1}_{zx_{i,j,k+1/2}}$ and $f^{n+1}_{zy_{i,j,k+1/2}}$ are obtained by using the standard Yee's FDTD algorithm for descretizing (13) and (14) as

$$\begin{aligned} &\left(f^{n+1}_{zx_{i,j,k+1/2}} - f^{n}_{zx_{i,j,k+1/2}}\right)\Big/\Delta t \\ &\quad + \sigma_x(i)\left(f^{n+1}_{zx_{i,j,k+1/2}} + f^{n}_{zx_{i,j,k+1/2}}\right)\Big/ 2\varepsilon_o \\ &= \sigma_x(i)\left(H^{n+1/2}_{y_{i+1/2,j,k+1/2}} - H^{n+1/2}_{y_{i-1/2,j,k+1/2}}\right)\Big/\Delta x\varepsilon_o \end{aligned} \tag{16}$$

$$\begin{aligned} &\left(f^{n+1}_{zy_{i,j,k+1/2}} - f^{n}_{zy_{i,j,k+1/2}}\right)\Big/\Delta t \\ &\quad + \sigma_y(j)\left(f^{n+1}_{zy_{i,j,k+1/2}} + f^{n}_{zy_{i,j,k+1/2}}\right)\Big/ 2\varepsilon_o \\ &= \sigma_y(j)\left(H^{n+1/2}_{x_{i,j+1/2,k+1/2}} - H^{n+1/2}_{x_{i,j-1/2,k+1/2}}\right)\Big/\Delta y\varepsilon_o. \end{aligned} \tag{17}$$

Equations (16) and (17) can be rearranged as

$$\begin{aligned} f^{n+1}_{zx_{i,j,k+1/2}} &= g_{zx_1}(i) f^{n}_{zx_{i,j,k+1/2}} \\ &\quad + g_{zx_2}(i)\left(H^{n+1/2}_{y_{i+1/2,j,k+1/2}} - H^{n+1/2}_{y_{i-1/2,j,k+1/2}}\right) \end{aligned} \tag{18}$$

$$\begin{aligned} f^{n+1}_{zy_{i,j,k+1/2}} &= g_{zy_1}(j) f^{n}_{zy_{i,j,k+1/2}} \\ &\quad + g_{zy_2}(j)\left(H^{n+1/2}_{x_{i,j+1/2,k+1/2}} - H^{n+1/2}_{x_{i,j-1/2,k+1/2}}\right) \end{aligned} \tag{19}$$

where

$$\alpha_x(i) = \frac{\Delta t \sigma_x(i)}{2\varepsilon_o}, \qquad g_{zx1}(i) = \frac{1-\alpha_x(i)}{1+\alpha_x(i)},$$

$$g_{zx2}(i) = \frac{2\alpha_x(i)/\Delta x}{1+\alpha_x(i)},$$

$$\alpha_y(j) = \frac{\Delta t \sigma_y(j)}{2\varepsilon_o}, \qquad g_{zy1}(j) = \frac{1-\alpha_y(j)}{1+\alpha_y(j)}$$

and

$$g_{zy2}(j) = \frac{2\alpha_y(j)/\Delta y}{1+\alpha_y(j)}.$$

Similar expressions can be obtained for the other field components.

It should be pointed out that the additional auxiliary variables, introduced in the above formulations, are zero outside the PML regions; therefore, they only need to be stored within the PML regions. In addition, (15) can be used for calculating the D_z field inside the FDTD domain by equating the additional auxiliary variables to zero.

Also, it should be mentioned that when the PML is used to truncate three dimensional FDTD domains, the application of the new formulations in the corner PML regions require only two additional auxiliary variables per field component per cell. In the face and the edge PML regions, simpler expressions can be obtained. As an example, for a wave propagating along the x-direction, the stretched coordinate variables in the x-face PML region are $S_x = 1 + \sigma_x/j\omega\varepsilon_o$, $S_y = S_z = 1$ [1]. Hence, the D_z field component is calculated from (15) with $f^{n+1}_{zy_{i,j,k+1/2}} = 0$. In this case, $f^{n+1}_{zx_{i,j,k+1/2}}$ is the only additional term to be stored.

III. Numerical Study

To validate the proposed method, numerical tests have been carried out in two dimensional FDTD domain for the TM case. The computational domain is chosen to be isotropic, homogeneous and with the size of $100\,\Delta x \times 50\,\Delta y$ cells where $\Delta x = \Delta y = 1.5$ cm is the space cell size in the x and y directions. The computational domain is excited by a point source at its center. The excitation used is similar to the derivative of the pulse used in [5]. This pulse was preferred as it has low numerical grid dispersion [6]. The time step used in the simulation is $\Delta t = 25$ ps and the reference FDTD solution, having no reflection errors form the domain boundaries, is calculated using a much larger computational domain ($400\,\Delta x \times 400\,\Delta y$).

The performance of the presented ADE–PML formulations was first investigated for lossless FDTD computational domain with the parameters $\varepsilon_r = \mu_r = 1$ and $\sigma = 0.0$. The computational domain was terminated by eight layer PML with different parameters: PML [8, 2, 0.01%], PML [8, 2, 0.001%], PML [8, 3, 0.01%], and PML [8, 3, 0.001%], as defined in Berenger's notation [1]. Figs. 1 and 2 show the local and the global errors, as

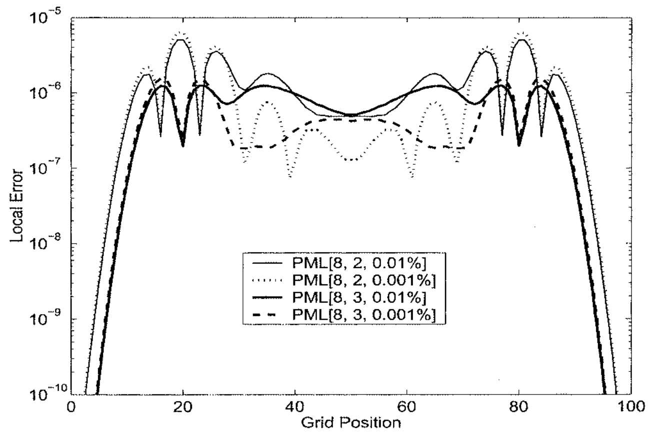

Fig. 1. Local error for the PML/computational domain interface along the line $(x, -25\,\Delta y)$ as observed at time $100\,\Delta t$ obtained using the ADE–PML for lossless FDTD domain ($\varepsilon_r = \mu_r = 1$ and $\sigma = 0.0$) and different PML parameters.

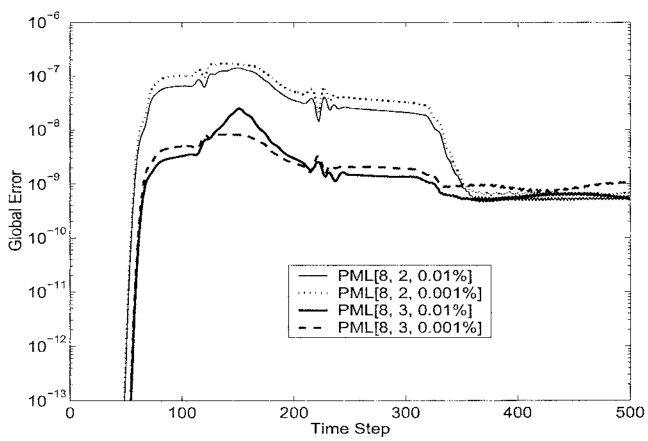

Fig. 2. Global error in the computational domain obtained using the ADE–PML for lossless FDTD domain ($\varepsilon_r = \mu_r = 1$ and $\sigma = 0.0$) and different PML parameters.

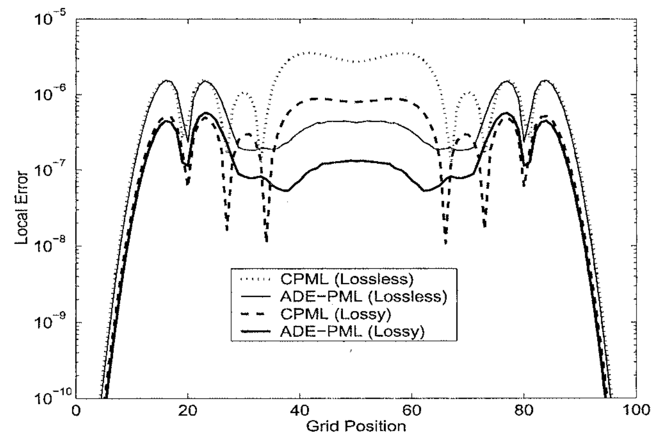

Fig. 3. Local error for the PML/computational domain interface along the line $(x, -25\Delta y)$ as observed at time $100\,\Delta t$ for PML[8, 3, 0.001%] obtained using the CPML and the ADE–PML for lossless ($\varepsilon_r = \mu_r = 1$ and $\sigma = 0.0$) and lossy ($\varepsilon_r = \mu_r = 1$ and $\sigma = 0.01$) FDTD domains.

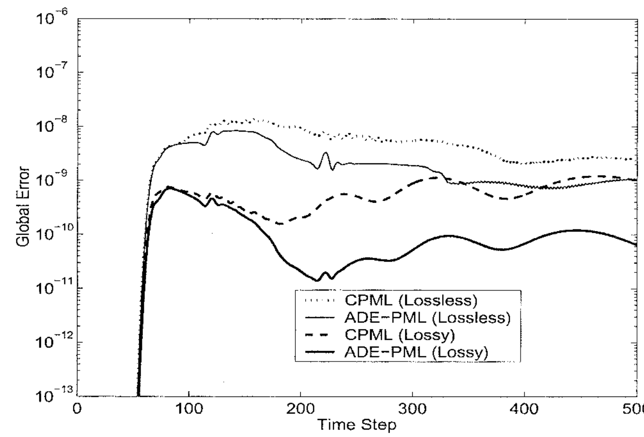

Fig. 4. Global error in the computational domain for PML[8, 3, 0.001%] obtained using the CPML and the ADE–PML for lossless ($\varepsilon_r = \mu_r = 1$ and $\sigma = 0.0$) and lossy ($\varepsilon_r = \mu_r = 1$ and $\sigma = 0.01$) FDTD domains.

defined in [5], for these numerical tests. The local error was calculated for the PML/computational domain interface along the line $(x, -25\,\Delta y)$ as observed at time $100\,\Delta t$. It can be observed that the presented ADE–PML formulations give good absorbing performance for all test cases. Also, it can be seen that PML [8, 3, 0.001%] gives the best results in these tests.

The performance of the ADE–PML was also compared with the CPML formulations for lossless ($\varepsilon_r = \mu_r = 1$, and $\sigma = 0.0$) and lossy ($\varepsilon_r = \mu_r = 1$, and $\sigma = 0.01$) FDTD domains. The FDTD domain is terminated by eight layer PML with the parameters PML[8, 3, 0.001%]. Figs. 3 and 4 show the local and the global errors for both cases. Two sets of results are presented: one for CPML and one for the ADE–PML. It can be observed from Figs. 3 and 4 that the ADE–PML gives better results compared with the CPML for the given PML parameters.

IV. Conclusion

A new algorithm based on the ADE method is presented for implementing the stretched coordinate PML formulations without the need of splitting the field components in the time domain. Simple and material independent PML formulations are obtained. Similar to the CPML and for truncating general media, the presented ADE–PML formulations require two additional auxiliary variables per field component in the PML region. Good absorbing performance has been observed for both lossless and lossy FDTD domains.

References

[1] J.-P. Berenger, “Three-dimensional perfectly matched layer for the absorption of electromagnetic waves,” *J. Comput. Phys.*, vol. 127, pp. 363–379, 1996.

[2] Q. H. Liu, “An FDTD algorithm with perfectly matched layer for conductive media,” *Microwave Opt. Technol. Lett.*, vol. 14, no. 2, pp. 134–137, 1997.

[3] J. A. Roden and S. D. Gedney, “Convolutional PML (CPML): An efficient FDTD implementation of the CFS-PML for arbitrary media,” *Microwave Opt. Technol. Lett.*, vol. 27, no. 5, pp. 334–339, 2000.

[4] A. P. Zhao, J. Juntunen, and A. V. Raisanen, “Material independent PML absorbers for arbitrary anisotropic dielectric media,” *Electron. Lett.*, vol. 33, no. 18, pp. 1535–1536, 1997.

[5] P. A. Tirkas, C. A. Balanis, and R. A. Renaut, “Higher order absorbing boundary conditions for the finite-difference time-domain method,” *IEEE Trans. Antennas Propagat.*, vol. 40, pp. 1215–1222, Oct. 1992.

[6] O. M. Ramahi, “Complementary operators: A method to annihilate artificial reflections arising from the truncation of the computational domain in the solution of partial differential equations,” *IEEE Trans. Antennas Propagat.*, vol. 43, pp. 697–704, July 1995.